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Boundary-layer solutions of single-mode convection equations

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Nonlinear thermal convection between two stress-free horizontal boundaries is studied using the modal equations for cellular convection. Assuming a large Rayleigh number R the boundary-layer method is used for different ranges of the Prandtl number σ. The heat flux F is determined for the values of the horizontal wavenumber a which maximizes F. For a large Prandtl number, $\sigma \gg R^{\frac{1}{6}}(\log R)^{-1}$, inertial terms are insignificant, a is either of order one (for $\sigma \geqslant R^{\frac{2}{3}}$) or proportional to $R^{\frac{1}{3}}\sigma^{-\frac{1}{2}}$ (for $\sigma \leqslant R^{\frac{2}{3}}$) and F is proportional to $R^{\frac{1}{3}}$. For a moderate Prandtl number,

$$(R^{-1}\log R)^{\frac{1}{5}} \ll \sigma \ll R^{\frac{1}{6}}(\log R)^{-1},$$

inertial terms first become significant in an inertial layer adjacent to the viscous buoyancy-dominated interior, and a and F are proportional to $R^{\frac{1}{4}}$ and

$$R^{\frac{3}{10}}\sigma^{\frac{1}{5}}(\log \sigma R^{\frac{1}{4}})^{\frac{1}{10}},$$

respectively. For a small Prandtl number, $R^{-1} \ll \sigma \ll (R^{-1}\log R)^{\frac{1}{5}}$, inertial terms are significant both in the interior and the boundary layers, and a and F are proportional to $(R\sigma)^{\frac{9}{32}}$ $(\log R\sigma)^{-\frac{1}{32}}$ and $(R\sigma)^{\frac{5}{16}}(\log R\sigma)^{\frac{3}{16}}$, respectively.

1. Introduction

We consider the effect of nonlinear momentum advection terms on thermal convection between two stress-free horizontal boundaries at large Rayleigh number. Our study is based on the so-called modal equations for momentum and heat. Briefly, these equations are constructed from the full Boussinesq equations by expanding the fluctuating quantities in a complete set of functions of the horizontal co-ordinates, and then truncating the expansion. For a more detailed discussion of these equations and their derivations, we refer to the paper by Gough, Spiegel & Toomre (1975, henceforth referred to as GST). The same system of equations had been derived earlier, differently, by Roberts (1966) using a procedure proposed by Glansdorff & Prigogine (1964).

In the problem of cellular convection considered in GST, the solutions for the single mode equations are derived by the boundary-layer method. The case in which the horizontal wavenumber a is of order one is mainly considered, but the boundary-layer solution for the case of large a is discussed briefly there. We find that some of the results in GST for the latter case appear to disagree with our results. Because of the importance of the nonlinear convection at various ranges of the Prandtl number, we found it useful to study the present problem. The disagreements with the results of GST are discussed in detail in § 4.

In the present study, we are interested in finding the solution which maximizes the heat transport F. The flow that maximizes F determines uniquely the horizontal wavenumber and gives an upper bound on the actual F. The success of the previous upper bound studies of thermal convection which exhibited interesting features, useful bounds on F and similarities with observation encouraged us to undertake the present study of the single mode equations. It is hoped that it will provide us with a deeper insight into the subject of the nonlinear convection at various values of the Prandtl number. The reader is also referred to GST for a detailed discussion in support of the studies based on the single-mode equations for cellular convection.

2. Governing equations

We consider an infinite horizontal layer of fluid depth d bounded above and below by two stress-free planes maintained at temperatures T_0 and $T_0 + \Delta T (\Delta T > 0)$, respectively. The modal equations for cellular convection are derived from the Boussinesq equations for momentum and heat by expanding the fluctuating variables in the plan form functions $f_n(x, y)$ of linear theory (GST). The non-dimensional steady state forms of these equations, after truncating the expansion by retaining only the first term, are:

$$(D^2 - a^2)^2 W = Ra^2\Theta + \frac{C}{\sigma}[2DW(D^2-a^2)W + W(D^2-a^2)DW], \tag{2.1a}$$

$$(D^2-a^2)\Theta + (1 - W\Theta + F)W = C(2WD\Theta + \Theta DW). \tag{2.1b}$$

In the above equations, W is the vertical dependence of the vertical component Wf_1 of the velocity vector $\mathbf{u}$, Θ is the vertical dependence of the deviation Θf_1 of the temperature from its horizontal average, $R = \alpha g \Delta T d^3/K\nu$ is the Rayleigh number, $\sigma = \nu/K$ is the Prandtl number, ν is the kinematic viscosity, α is the coefficient of thermal expansion, K is the thermal diffusivity, and g is the acceleration due to gravity. Also, a is the horizontal wavenumber, $D = d/dz$, $F = \langle W\Theta \rangle$ is the heat flux, $C = \frac{1}{2}\overline{(f_1(x,y))^3}$ is the parameter derived from the planform function $f_1(x,y)$. The bars denote horizontal average, and the angle brackets denote a further vertical average over the whole layer. The constant C vanishes for rolls and rectangles and takes the value of $6^{-\frac{1}{2}}$ for the hexagonal planform. We shall assume $C \neq 0$ and consider the value such as $6^{-\frac{1}{2}}$ as representative value of C. For $C = 0$, the system (2.1) reduces to the so-called mean field equations, and the problem has been solved and discussed by Howard (1965), Roberts (1966) and others.

We shall rescale our dependent variables such that

$$\omega = (FR)^{-\frac{1}{2}} W, \quad \theta = (R/F)^{\frac{1}{2}} \Theta. \tag{2.2}$$

The governing equations then take the following forms:

$$(D^2-a^2)^2\omega = a^2\theta + \left(\frac{C}{\sigma}\right)(FR)^{\frac{1}{2}}[2D\omega(D^2-a^2)\omega + \omega(D^2-a^2)D\omega], \tag{2.3a}$$

$$\frac{1}{FR}(D^2-a^2)\theta + \left(1 - \omega\theta + \frac{1}{F}\right)\omega = C(FR)^{-\frac{1}{2}}(2\omega D\theta + 0D\omega). \tag{2.3b}$$

The constraint

$$F = \frac{1 - R^{-1}\langle|\nabla\theta|^2\rangle}{\langle(1-\omega\theta)^2\rangle}, \tag{2.4}$$

is obtained by multiplying (2.3*b*) by θ and taking the vertical average over the layer which is used to evaluate F. The boundary conditions to be considered for the free surfaces at $z = 0, 1$ are

$$\omega = D^2\omega = \theta = 0. \tag{2.5}$$

The subsequent analysis and solution of (2.3)–(2.5) supposes throughout that both the Rayleigh number and the heat flux are large. Different classes of solutions are found for different orders of magnitude of the Prandtl number σ. In each case, the principal focus is on the unique solution that maximizes F.

3. Solution by boundary-layer method

3.1. *The case of a large Prandtl number*

The solution in the range $\sigma \geqslant R^{\frac{2}{3}}$ is essentially that given by Howard (1965) for $C = 0$. The inertial terms are insignificant in this range and the boundary layer is not affected by σ. The boundary-layer structure consists of a nonuniform interior and a thin thermal layer of thickness δ close to each boundary. In the interior of the layer, viscous, buoyancy and convection terms are significant, and the dependent variables are of order one. In the thermal layer, viscous, conduction and convection terms are important and we find that $\omega \sim \delta$ and $\theta \sim \delta^{-1}$. The heat flux is independent of σ and is maximized for the wavenumber a which is found to be of order one. The dependence of F and δ on R is the same as in the case of $C = 0$. That is, $F = 0{\cdot}325(1+C^2)^{-\frac{1}{3}}R^{\frac{1}{3}}$ and $\delta = 1{\cdot}449(1+C^2)^{\frac{1}{12}}R^{-\frac{1}{3}}$.

The solution in the range $R^{\frac{1}{3}}(\log R)^{-1} \ll \sigma \ll R^{\frac{2}{3}}$ is qualitatively the same as in the range $\sigma \geqslant R^{\frac{2}{3}}$, except that a is now in the maximizing range $a = O(R^{\frac{1}{3}}\sigma^{-\frac{1}{2}})$. Since a is now large, there exists also an intermediate layer of thickness a^{-1}. The interior of the layer is now uniform and we find that $\omega \sim a^{-1}$ and $\theta \sim a$ in this region and in the intermediate layer. In the thermal layer, $\omega \sim \delta$ and $\theta \sim \delta^{-1}$. The expressions for F and δ are now: $F = (2{\cdot}124)^{-\frac{4}{3}}(1+C^2)^{-\frac{1}{3}}R^{\frac{1}{3}}$ and $\delta = (2{\cdot}124)^{\frac{1}{3}}(1+C^2)^{\frac{1}{12}}R^{-\frac{1}{3}}$.

3.2. *The case of a moderate Prandtl number*

The wavenumber a is supposed to be large (which can be justified *a posteriori*), so that the convection cells are narrow. The solutions can be obtained by matching asymptotic approximations in the interior and three distinct regions near each boundary. Without loss of generality, we shall restrict ourselves to the discussion of the boundary-layer structure near the lower boundary, since the boundary-layer structure near the upper boundary is essentially the same as the one near the lower boundary.

In the interior of the layer, z is of order one. It is assumed that

$$a^4 \ll FR \ll a^6\sigma^2. \tag{3.1}$$

The governing equations (2.3*a*)–(2.3*b*) yield, after using (3.1), the following equations

$$a^2\omega = 0, \quad \omega 0 = 1. \tag{3.2a, b}$$

It is seen from (3.2) that the viscous, buoyancy and convection terms are important in the interior. Equations (3.2a, b) are satisfied by

$$\omega = a^{-1}, \quad \theta = a. \tag{3.3a, b}$$

Near the boundary and adjacent to the interior is an inertial layer in which inertial terms are significant. We define ϵ as the thickness of the layer and $\zeta = z/\epsilon$ as the variable in the layer. We then find from (2.3), after applying matching conditions (matching the solutions to the corresponding solutions in the interior), that the equations in the inertial layer are

$$a^2\omega = \theta - 3a^3\omega\frac{d\omega}{d\zeta}, \quad \omega\theta = 1, \tag{3.4a, b}$$

where it is found appropriate to assume that

$$\epsilon = C\sigma^{-1}a^{-3}(FR)^{\frac{1}{2}} \gg a^{-1}. \tag{3.5}$$

The solution to (3.4a) satisfying the boundary condition $\omega = 0$ at $\zeta = 0$ satisfies the following equation

$$-3a\omega + \tfrac{3}{2}\log\left(\frac{1+a\omega}{1-a\omega}\right) = \zeta. \tag{3.6}$$

(3.4b) and (3.6) yield the following asymptotic results

$$\omega = a^{-1}\zeta^{\frac{1}{3}}, \quad \theta = a\zeta^{-\frac{1}{3}} \quad \text{as} \quad \zeta \to 0. \tag{3.7a, b}$$

Closer to the boundary and adjacent to the inertial layer is an intermediate layer of thickness a^{-1}, in which vertical derivatives are important in the inertial terms. Defining $\xi = az$ as the variable in this layer, the equations (2.3) and matching conditions (matching the solutions to the corresponding solutions in the inertial layer) yield

$$\theta + C\sigma^{-1}a(FR)^{\frac{1}{2}}\left[2\frac{d\omega}{d\xi}\left(\frac{d^2}{d\xi^2} - 1\right)\omega + \omega\left(\frac{d^2}{d\xi^2} - 1\right)\frac{d\omega}{d\xi}\right] = 0, \tag{3.8a}$$

$$\omega\theta = 1. \tag{3.8b}$$

Solutions of (3.8a)–(3.8b) are given by

$$\omega = \left(\frac{\sigma}{Ca}\right)^{\frac{1}{3}}(FR)^{-\frac{1}{6}}\,\xi(3\log\xi^{-1})^{\frac{1}{3}} \quad \text{as} \quad \xi \to 0, \tag{3.9a}$$

$$\theta = \left(\frac{Ca}{\sigma}\right)^{\frac{1}{3}}(FR)^{\frac{1}{6}}\,\xi^{-1}(3\log\xi^{-1})^{-\frac{1}{3}} \quad \text{as} \quad \xi \to 0. \tag{3.9b}$$

There is a further thinner layer closer to the boundary, in which thermal conduction is significant in the heat equation and θ is brought to its zero boundary value. We define δ as the thickness of the layer and $\eta = z/\delta$ as the variable in the layer. The governing equations and matching conditions then give the following equations in the thermal layer:

$$\frac{d^4\omega}{d\eta^4} = \sigma^{-1}C\delta(FR)^{\frac{1}{2}}\left(2\frac{d\omega}{d\eta}\cdot\frac{d^2\omega}{d\eta^2} + \omega\frac{d^3\omega}{d\eta^3}\right), \tag{3.10a}$$

$$\frac{1}{FR\,\delta^2}\frac{d^2\theta}{d\eta^2} + (1-\omega\theta)\,\omega = C\delta^{-1}(FR)^{-\frac{1}{2}}\left(2\omega\frac{d\theta}{d\eta} + \theta\frac{d\omega}{d\eta}\right). \tag{3.10b}$$

In deriving (3.10), it is found that we must have the following conditions

$$FRA^2\delta^2 = 1, \quad a\delta \ll 1, \quad \sigma a^2\delta^4 \ll A^2, \tag{3.11a, b, c}$$

where

$$A = \left(\frac{\sigma}{Ca}\right)^{\frac{1}{3}} (FR)^{-\frac{1}{6}}\, a\delta \left(3 \log \frac{1}{a\delta}\right)^{\frac{1}{3}}. \tag{3.11d}$$

The solutions to (3.10) satisfying (2.5) are

$$\omega = A\eta, \quad \theta = \frac{\eta C^{\frac{1}{2}}}{2A} \int_1^{\mu^2} (\mu^2 - t^2)^{-\frac{1}{4}} \exp\left[\frac{c}{2}\eta^2(1-t)\right] dt, \tag{3.12a, b}$$

where

$$\mu^2 = 1 + C^{-2}. \tag{3.12c}$$

To determine F, we must evaluate the expressions $\langle|\nabla\theta|^2\rangle$ and $\langle(1-\omega\theta)^2\rangle$ in (2.4). Within the boundary-layer approximation, using the results obtained above and keeping only the leading-order terms, we find that

$$\langle|\nabla\theta|^2\rangle = a^4 + 2\delta^{-1}A^{-2}I_1, \tag{3.13a}$$

$$\langle(1-\omega\theta)^2\rangle = 2\delta I_2, \tag{3.13b}$$

where I_1 and I_2 are the integrals

$$\int_0^\infty (d\theta/d\eta)^2\, d\eta \text{ and} \int_0^\infty (1-\eta\theta)^2\, d\eta$$

in the thermal layer, respectively. Using (3.13) in (2.4) and maximizing F with respect to a, yield the following results

$$\epsilon = \left(\frac{39}{5}\right)^{\frac{1}{10}} \left(\frac{6}{I}\right)^{\frac{3}{5}} \left(\frac{C}{\sigma}\right)^{\frac{9}{10}} R^{-\frac{1}{10}} (\log \sigma R^{\frac{1}{4}})^{\frac{1}{10}}, \tag{3.14a}$$

$$a = \left(\frac{R}{13}\right)^{\frac{1}{4}}, \tag{3.14b}$$

$$\delta = \left(\frac{5I}{18}\right)^{\frac{1}{5}} (13)^{\frac{3}{10}} \left(\frac{C}{\sigma}\right)^{\frac{1}{5}} R^{-\frac{3}{10}} (\log \sigma R^{\frac{1}{4}})^{-\frac{1}{5}}, \tag{3.14c}$$

$$F = \left(\frac{6}{13I}\right)^{\frac{6}{5}} \left(\frac{3}{5}\right)^{\frac{1}{5}} (13)^{-\frac{1}{10}} \left(\frac{\sigma}{C}\right)^{\frac{1}{5}} R^{\frac{3}{10}} (\log \sigma R^{\frac{1}{4}})^{\frac{1}{5}}, \tag{3.14d}$$

where it is found that

$$I = I_1 + I_2 = 1{\cdot}062(1 + C^2)^{\frac{1}{4}}. \tag{3.14e}$$

Various assumptions including (3.1), (3.5) and (3.11) lead us to the following conditions for the validity of the solutions

$$(R^{-1} \log R)^{\frac{1}{6}} \ll \sigma \ll R^{\frac{1}{8}} (\log R)^{-1}. \tag{3.15}$$

3.3. *The case of a small Prandtl number*

The wavenumber a is again supposed to be large (which can be justified *a posteriori*). The boundary-layer structure for this case consists of a non-uniform interior and two distinct regions near each boundary. In the interior of the layer, z is of order one. It is assumed that

$$a^4 \ll FR\sigma, \quad a^6\sigma^2 \ll FR. \tag{3.16a, b}$$

Using (3.16), the governing equations (2.3*a*)–(2.3*b*) yield

$$\theta = 3C\sigma^{-1}(FR)^{\frac{1}{2}}\,\omega D\omega, \quad \omega\theta = 1. \tag{3.17a, b}$$

It is seen from (3.17) that the inertial, buoyancy and convection terms are important in the interior. (3.17*a*, *b*) are satisfied by

$$\omega = \left(\frac{\sigma}{C}z\right)^{\frac{1}{3}}(FR)^{-\frac{1}{6}}, \quad \theta = \left(\frac{\sigma}{C}z\right)^{-\frac{1}{3}}(FR)^{\frac{1}{6}}, \tag{3.18a, b}$$

where the constant of integration is chosen so that ω satisfies its boundary condition at $z = 0$. Near each surface and adjacent to the interior are intermediate layers of thickness a^{-1}, in which vertical derivatives are important in the inertial terms. Defining $\xi_t = a(1-z)$ and $\xi = az$ as the variables in the upper and lower layers, respectively, the governing equations and matching conditions yield the following equations in the upper layer:

$$2\frac{d\omega}{d\xi_t}\left(\frac{d^2}{d\xi_t^2}-1\right)\omega+\omega\left(\frac{d^2}{d\xi_t^2}-1\right)\frac{d\omega}{d\xi_t} = 0, \quad \omega\theta = 1. \tag{3.19a, b}$$

Similarly, the governing equations yield (3.8) in the lower layer. Equation (3.8) yield (3.9) and (3.19*a*, *b*) yield the following results

$$\omega = \left(\frac{\sigma}{2C}\right)^{\frac{1}{3}}(FR)^{-\frac{1}{6}}(3\xi_t)^{\frac{2}{3}} \quad \text{as} \quad \xi_t \to 0, \tag{3.20a}$$

$$\theta = \left(\frac{2C}{\sigma}\right)^{\frac{1}{3}}(FR)^{\frac{1}{6}}(3\xi_t)^{-\frac{2}{3}} \quad \text{as} \quad \xi_t \to 0. \tag{3.20b}$$

Closer to each surface and adjacent to the intermediate layers are thermal layers. We define δ_t and δ as the thicknesses of the top and bottom layers, respectively. Also, $\eta_t = (1-z)/\delta_t$ and $\eta = z/\delta$ are defined to be the corresponding variables in these layers. We then find from (2.3), after applying matching conditions, that the equations in the lower thermal layer are (3.10) and in the upper thermal layer are

$$2\frac{d\omega}{d\eta_t}\cdot\frac{d^2\omega}{d\eta_t^2}+\omega\frac{d^3\omega}{d\eta_t^3} = 0, \tag{3.21a}$$

$$(FR\delta_t^2)^{-1}\frac{d^2\theta}{d\eta_t^2}+(1-\omega\theta)\,\omega+C\delta^{-1}(FR)^{-\frac{1}{2}}\left(2\omega\frac{d\theta}{d\eta_t}+\theta\frac{d\omega}{d\eta_t}\right) = 0, \tag{3.21b}$$

and it is found necessary to require conditions (3.11) and

$$FRA_t^2\delta_t^2 = 1, \quad a\delta_t \ll 1, \quad \sigma a^2\delta_t^4 \ll A_t^2, \tag{3.22a, b, c}$$

where

$$A_t = \left(\frac{\sigma}{2C}\right)^{\frac{1}{3}}(FR)^{-\frac{1}{6}}(3a\delta_t)^{\frac{2}{3}}. \tag{3.22d}$$

The solutions to (3.10) and (3.21) are (3.12) and

$$\omega = A_t\eta_t, \tag{3.23a}$$

$$\theta = -\frac{C^{\frac{1}{2}}\eta_t}{2A_t}\int_1^{\mu^2}(\mu^2-t^2)^{-\frac{1}{4}}\exp\left[\frac{C}{2}\eta_t^2(t-1)\right]dt. \tag{3.23b}$$

The maximization of F proceeds as before, and we find

$$a = (3)^{-\frac{1}{2}}(32)^{\frac{1}{32}}(7)^{-\frac{15}{32}}\left(\frac{7I}{6}\right)^{\frac{3}{16}}\left(\frac{R\sigma}{C}\right)^{\frac{9}{32}}(\log R\sigma)^{-\frac{1}{32}}, \tag{3.24a}$$

$$\delta_t = (48)^{-\frac{1}{5}}\left(\frac{7I}{6}\right)^{\frac{3}{20}}(224)^{\frac{9}{40}}\left(\frac{R\sigma}{C}\right)^{-\frac{3}{8}}(\log R\sigma)^{-\frac{1}{40}}, \tag{3.24b}$$

$$\delta = \left(\frac{7I}{6}\right)^{\frac{1}{8}}(224)^{\frac{3}{16}}\left(\frac{R\sigma}{C}\right)^{-\frac{5}{16}}(\log R\sigma)^{-\frac{3}{16}}, \tag{3.24c}$$

$$F = \left(\frac{6}{7I}\right)^{\frac{9}{8}}(224)^{-\frac{3}{16}}\left(\frac{R\sigma}{C}\right)^{\frac{5}{16}}(\log R\sigma)^{\frac{3}{16}}. \tag{3.24d}$$

Various assumptions including (3.1), (3.11) and (3.22) lead us to the following condition for the validity of the solutions

$$R^{-1} \ll \sigma \ll (R^{-1}\log R)^{\frac{1}{5}}. \tag{3.25}$$

4. Discussion

The boundary-layer analysis has shown that it is appropriate to divide the parameter space into four different regions. For $\sigma \geqslant R^{\frac{2}{3}}$, the interior and the boundary-layer regions are unaffected by σ. F is maximized by a value of a which is of order one and F_{max} is independent of σ. The effect of the inertial terms is sufficiently small such that the maximizing flow is essentially identical to that at infinite Prandtl number. For $R^{\frac{1}{6}}(\log R)^{-1} \ll \sigma \ll R^{\frac{2}{3}}$, the inertial terms are still insignificant, as far as F_{max} is concerned. Although F_{max} is independent of σ, the flux-maximizing value of a depends strongly on σ. The horizontal scale of motion is fixed by the flux-maximizing value of a, once the inertial terms balance the viscous and buoyancy terms in the intermediate layer. The upper and lower limits on σ are determined by balancing the inertial terms with viscous terms in the intermediate layer and by using the fact that F_{max} is unaffected by the interior solutions. For $(R^{-1}\log R)^{\frac{1}{5}} \ll \sigma \ll R^{\frac{1}{6}}(\log R)^{-1}$, F_{max} is an increasing function of σ, but the flux-maximizing value of a is independent of σ. The order of magnitude of F_{max} in this range is always less than its values in the above first two regions. In the thermal layers, the inertial term is important for $\sigma \ll 1$ and the viscous term is important for $\sigma \gg 1$. The boundary-layer structure near the lower boundary is essentially the same as the one near the upper boundary mainly because of the uniformity of the interior. The upper and lower limits on σ are determined from various conditions including (3.1), (3.5) and (3.11). For $R^{-1} \ll \sigma \ll (R^{-1}\log R)^{\frac{1}{5}}$, F_{max} and the flux-maximizing value of a are both increasing functions of σ. The interior is now non-uniform. The intermediate and thermal layers near the lower boundary are essentially the same as the corresponding ones in the previous case. The boundary-layer structure near the upper boundary is now different from the one near the lower boundary. It should be realized of course that a corresponding solution exists for which the designations 'upper' and 'lower' are interchanged. The value of a which maximizes F is determined essentially by the contribution of the interior solution to the conduction term in the expression (2.4) for F. It depends strongly on σ, since inertial terms are now significant in the interior and affect the relations there. The upper and lower limits on σ are found from various assumptions including (3.1), (3.11) and (3.22). An interesting qualitative result of the present boundary-layer analysis is that F_{max} and the flux-maximizing a are continuous functions (within a logarithmic term) of R and σ throughout the range $\sigma \gg R^{-1}$.

In the studies of modal equations for cellular convection presented in GST the case of large wavenumber convection in a layer with stress-free boundaries is briefly discussed. The main results of GST for this case are summarized here. For $\sigma \gg (Ra^2)^{\frac{2}{3}}$, the boundary-layer structure consists of a non-uniform interior and a thermal layer near each boundary. F is maximized for the value of a which is of order one and $F_{\max}$ is proportional to $R^{\frac{1}{3}}$. For $R^{-1} \ll \sigma \ll (Ra^2)^{\frac{2}{3}}$, a similar boundary-layer structure exists, but the boundary layer near the upper boundary is now different from the one near the lower boundary. F is maximized for the value of a which is proportional to $R^{\frac{1}{4}}$ and $F_{\max} \propto R^{\frac{3}{10}}\sigma^{\frac{1}{5}}(\log R^{\frac{3}{2}}\sigma)^{\frac{1}{5}}$. For $\sigma \ll R^{-1}$, heat is transported mainly by conduction and $F = \Lambda(R\sigma/C)^2$, where Λ is a function of a, which attains its maximum value $(2{\cdot}496)\,10^{-6}$ at $a = 2{\cdot}37$. Using the value of the flux-maximizing wavenumber in each of the first two ranges for σ, we find that both of these boundary-layer structures can similtaneously exist for $R^{\frac{2}{3}} \ll \sigma \ll R$. There is a further undesirable result. For $\sigma \sim (Ra^2)^{\frac{2}{3}}$, or $\sigma \sim R^{-1}$, there exists discontinuities in $F_{\max}$ and the flux-maximizing value of a. Comparing these results with those of the present study, we find the following conclusions. For $\sigma \geqslant R^{\frac{2}{3}}$, our main results are qualitatively equivalent to those in GST for $\sigma \gg (Ra^2)^{\frac{2}{3}}$. For $R^{\frac{1}{6}}(\log R)^{-1} \ll \sigma \ll R^{\frac{2}{3}}$, our results disagree with those in GST. For $(R^{-1}\log R)^{\frac{1}{6}} \ll \sigma \ll R^{\frac{1}{6}}(\log R)^{-1}$, our boundary-layer structure, solutions for vertical velocity and temperature and the expression for $F_{\max}$ all disagree with those in GST. The flux-maximizing value of a is, however, the same as that in GST. For $R^{-1} \ll \sigma \ll (R^{-1}\log R)^{\frac{1}{6}}$, our boundary-layer structure and solutions for vertical velocity and temperature appear to be equivalent to those in GST for

$$R^{-1} \ll \sigma \ll (Ra^2)^{\frac{2}{3}},$$

but $F_{\max}$ and the flux-maximizing value of a disagree. As $\sigma \to R^{-1}$, $F_{\max}$ becomes $O(1)$ and can no longer be assumed large. For $\sigma \ll R^{-1}$, our results (though not given in this paper) agree with those in GST. In contrast to the results in GST, we do not have discontinuities in the expressions for $F_{\max}$ and the flux-maximizing value of a, and the boundary-layer structures do not overlap.

The nonlinear Bénard convection contains three different types of nonlinearities: nonlinear interactions of the fluctuating velocity with the mean temperature gradient (referred to as WT), nonlinear interactions of the fluctuating velocities in the momentum equation (referred to as WW) and deviation of WT from the nonlinear advection of temperature in the heat equation (referred to as $W\Theta$). Our present results, for $\sigma \gg R^{-1}$, indicate that WT is significant in both interior and the boundary layer regions and affects the solutions qualitatively. WW is significant in the boundary-layer regions, for a moderate or small σ (except in the thermal layer for $\sigma \gg 1$) and in the interior (for a small σ), and it affects the solutions qualitatively for a moderate or small σ. $W\Theta$ is significant only in the thermal layers and affects the solutions quantitatively. These results support the general belief that WW is relatively small whenever σ is large, and $W\Theta$ can be ignored whenever the qualitative features of the convection problem are concerned. The latter statement is not quite obvious, but it is expected to hold as far as heat transport processes are concerned.

Finally, we present a critique of the modal approach which is especially desirable in view of its rather uncritical acceptance by various authors (Spiegel 1971; GST; Toomre, Gough & Spiegel 1977; Gough 1977) who have also given detailed discussions in favour of the studies based on the modal equations. The modal equations represent a special approximation of the full Boussinesq equations in that a hexagonal symmetry of the horizontal dependence is presumed. It yields the undesirable results of a non-symmetric z-dependence of the solutions, once it is applied to a symmetric layer. There is also the possibility that preferred convective motion can not be described adequately by the modal equations. In particular, some solutions of the full equations may exhibit a higher heat transport than those of the modal equations. The so-called 'flywheel' solutions discovered first by Jones, Moore & Weiss (1976) and more recent two-dimensional studies along the same line (Clever & Busse 1981); Busse & Clever 1981) represent some challenge to the solutions based on the modal equations for a low σ. The asymmetry in the treatment of vertical and horizontal dependence in the modal equations of convection do indeed prevent the representation of solutions for which nonlinearities in the horizontal dependence of the problem is important. So far the study of the flywheel solutions has been restricted to the two-dimensional steady case. Although these solutions are known to be unstable to oscillatory instability, but it is quite likely that time dependent three-dimensional solutions with similar properties as the fly wheel solutions do exist and are responsible for high heat transport in a low-Prandtl-number fluid. The modal equations are adequate only in cases where the inherent time dependent turbulent convection does not cause large deviations in the average properties from the corresponding steady solutions. Another point concerning our single-mode approach is that convective flow having a moderate or small horizontal length scale is predicted, a result which is not supported by the experimental evidence. The multi-scales character of the real flow at large R also suggests that multi-modal solutions (Busse 1969) which allow greater heat flux than single-mode solutions and are characterized by several length scales are likely to be preferred at sufficiently large R.

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